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Characterizing Herdability of Signed Networks via Graph Walks

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***Abstract*—Herdability, as a variant of controllability, indicates the capability of driving system states to a specific subset of the state space. This paper investigates the topological characterizations of the herdability of signed networked systems. Specifically, a dynamic signed leader-follower network is considered, where a small subset of the network nodes (i.e., the leaders) are endowed with exogenous control input and the rest nodes are influenced by the leaders via the underlying network connectivity. The considered network allows positive and negative edges to capture cooperative and competitive interactions, thus resulting a signed graph. Motivated by the practical applications, the systems states are required to be driven by the leaders to be element-wise larger than a positive threshold, i.e., a specific subset rather than the entire state space as in the classical controllability. To study the herdability of signed leader-follower networks, this work extends the results of [1] and characterizes topological structures of herdable networks. In particular, graph walk is exploited to develop sufficient conditions ensuring the herdability of signed networks via 1-walks and 2-walks. These results are then extended to acyclic graphs with multiple walks. Examples are provided to illustrate the developed topological characterizations.**

I. INTRODUCTION

Networked systems have broad applications in social science, biological science, and engineering [2]–[5]. In such applications, network controllability has long been a research focus, since a fully controllable system indicates that the system states can be driven to any desired state via controls. However, requiring a system to be fully controllable is often too restrictive, and unnecessary in many practical applications. For example, connected autonomous vehicles are often required to maintain above a desired positive speed in cruise control. In political election, a candidate wins if the voting rate that is supportive is sufficiently larger than a positive percentage. In these applications, fully controllable systems become unnecessary, since driving the vehicles' speed or participants' voting rate to be negative does not make any physical sense. Instead, the relaxed controllability that the system states can be driven to a specific subset, rather than the entire state space as in controllability, is of more practical significance. Such a relaxed controllability is referred to as herdability [1]. To this end, this work is practically motivated to investigate the topological characterizations of the herdability of networked systems.

Since herdability is closely related to controllability, the literature review of controllability is discussed. Based on the type of interactions, network can be classified as either cooperative or non-cooperative networks. Cooperative networks are often modeled as unsigned graphs allowing only positive edge weights to represent cooperative interactions between network components, while non-cooperative networks are often modeled as signed graphs allowing positive and negative edge weights to represent cooperative and competitive interactions. Complete controllability of cooperative networks has been extensively studied in the literature. For instance, graph theoretic approaches [6]–[8] and structural controllability [9]–[11] were explored to reveal how network topological structures influence the controllability via the leader group by external controls. Consensus based results were also investigated to characterize the network controllability in the works of [12]–[14]. When considering non-cooperative networks, controllability of signed graphs were studied via structural balance in the works of [15]–[19]. However, the aforementioned results mainly considered the characterization of complete controllability. Topological characterizations of herdability remain largely unknown.

Compared to classical controllability, herdability indicates the capability of driving system states to a specific subset in the state space. Herdability was first studied in the recent work [1], where the system states are controlled to be element-wise larger than a non-negative threshold. The impact of the topological structure on the herdability of positive systems was investigated. Positive system is a particular class of systems where, provided positive initial states, the states remain positive during evolution [20]. Herdability of positive systems was recently investigated in the works of [1] and [21]. Necessary and sufficient conditions on herdable positive networked system were developed in [1]. The results of [1] were then extended to characterize herdability for positive complex networks in [21].

Inspired by the works of [1] and [21], this work considers the herdability of a signed networked system. Specifically, we consider a dynamic signed leader-follower network, where a small subset of the network nodes (i.e., the leaders) are endowed with exogenous control input and the rest nodes are influenced by the leaders via the underlying network connectivity. The considered network allows positive and negative edges to capture cooperative and competitive interactions, thus resulting a signed graph. Motivated by the

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practical applications, the systems states are required to be driven by the leaders to be element-wise larger than a positive threshold, i.e., a specific subset rather than the entire state space as in the classical controllability. To study the herdability of signed leader-follower networks, graph walk is exploited to develop sufficient conditions ensuring the herdability of signed networks via 1-walks, 2-walks. These results are then extended to acyclic graphs with multiple walks. Compared to [1] and [21] where herdability was investigated for positive systems, this work mainly focuses on the characterization of herdability of general signed networked systems. Examples are provided to illustrate the developed topological characterizations.

II. Problem Formulation

Consider a network modeled by a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ denotes the node set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the edge set. The interactions between nodes are captured by the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} \neq 0$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. No self-loop is considered, i.e., $a_{ii} = 0$, $\forall i = 1, \ldots, n$. The weight $a_{ij} \in \mathbb{R}$ is allowed to take real numbers, where $a_{ij} \in \mathbb{R}^+$ and $a_{ij} \in \mathbb{R}^-$ represent cooperative and competitive interactions between node v_i and v_j in the network, respectively. Throughout the rest of this work, an edge (v_i, v_j) is called positive if $a_{ij} \in \mathbb{R}^+$, and negative otherwise. The ith row and jth column of $\mathcal{A}$ are denoted by $\mathcal{A}_{i,:}$ and $\mathcal{A}_{:,j}$, respectively. The ith row to the kth row of the column j of $\mathcal{A}$ is denoted by $\mathcal{A}_{i:k,j}$.

Let $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ denote the stacked system states[1] of the network $\mathcal{G}$, where each entry $x_i(t) \in \mathbb{R}$ represents the state of node v_i. It is assumed that a subset $\mathcal{V}_l \subseteq \mathcal{V}$ of m nodes, referred as leaders in the network, can be endowed with external controls. The rest nodes $\mathcal{V}_f = \mathcal{V} \setminus \mathcal{V}_l$ are referred as followers with $\mathcal{V}_l \cap \mathcal{V}_f = \emptyset$. Without loss of generality, the leaders' and the followers' indices are assumed to be $\mathcal{V}_l = \{1, \ldots, m\}$ and $\mathcal{V}_f = \{m+1, \ldots, n\}$. Suppose the system states evolve according to the linear dynamics

$$\dot{x}(t) = \mathcal{A}x(t) + Bu(t), \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n}$ is the adjacency matrix and $B = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \in \mathbb{R}^{n\times m}$ is the input matrix with basis vectors e_i, $i = 1, \ldots, m$, indicating that the ith node is endowed with external controls $u(t) \in \mathbb{R}^m$.

The herdability of the system in (1) is defined as follows.

Definition 1 (Network Herdability [1])**.** A networked system with dynamics in (1) is herdable if there exists a control input $u(t)$ that can drive $x(t)$ to the set $H_d = \{x(t) : x_i(t) \geq d, \forall v_i \in \mathcal{V}\}$, where d is an arbitrary positive threshold.

Definition 1 indicates that a network is herdable if its states can be driven to a specific subset (i.e., H_d) of the state space. Recall that the controllability matrix $\mathcal{C} = \begin{bmatrix} B & \mathcal{A}B & \cdots & \mathcal{A}^{n-1}B \end{bmatrix}$ indicates the controllable subspace of a system [22]. The system in (1) is completely controllable, if the controllability matrix $\mathcal{C}$ has full row rank. Since this work concerns driving $x(t)$ to H_d, a subset of the entire state space, the following lemma shows how the herdability of a system relates to the controllability matrix.

Lemma 1. *[1] A networked system with dynamics in (1) is herdable to the set H_d if and only if there exists an element-wise positive vector $k \in \mathrm{range}(\mathcal{C})$, where $\mathrm{range}(\cdot)$ represents the range space of a matrix.*

As indicated in Lemma 1, the herdability depends on the range space of the controllability matrix $\mathcal{C}$. Motivated by this observation, the objective of this work is to develop topological characterizations of herdable networks based on Lemma 1.

III. Main Results

A. Graph Walks

Graph walks will be used as main tools to develop topological characterizations of network herdability. Graph walk is defined as an alternating sequence of nodes and edges [23]. Let $\alpha(v_i, v_j)$ denote a walk from v_i to v_j on graph $\mathcal{G}$, i.e., $\alpha(v_i, v_j) = v_i, (v_i, v_m), v_m, \ldots, v_p, (v_p, v_j), v_j$, with nodes and edges belonging to $\mathcal{V}$ and $\mathcal{E}$. Since $\mathcal{G}$ is undirected, $\alpha(v_i, v_j)$ also indicates that v_j is reachable from v_i via the walk. It is worth pointing out that graph walk may have repeated edges, which is different from the path of a graph that comprises only distinct edges [23]. For instance, the path from v_1 to v_2 in Fig. 1 is $v_1, (v_1, v_2), v_2$, while a walk from v_1 to v_2 could be $\alpha(v_1, v_2) = v_1, (v_1, v_2), v_2, (v_2, v_1), v_1, (v_1, v_2), v_2$, which includes repetitive edges.

The length of a walk $\alpha(v_i, v_j)$ is defined as its number of edges along the walk, including repetitive edges. Let $\alpha^{(k)}(v_i, v_j)$ denote a k-walk which indicates that there is a walk of length k from v_i to v_j. Let $\eta^{(k)}(v_i, v_j)$ denote the set that contains all k-walks from v_i to v_j. The weight of a k-walk, denoted by $w\left(\alpha^{(k)}(v_i, v_j)\right) \in \mathbb{R}$, is the product of the edge weights in the walk of $\alpha^{(k)}(v_i, v_j)$. Let $\mathrm{sign}\left(w\left(\alpha^{(k)}(v_i, v_j)\right)\right)$ denote the sign of the k-walk from v_i to v_j, where $\mathrm{sign}(\cdot)$ is the sign function. If $\mathrm{sign}\left(w\left(\alpha^{(k)}(v_i, v_j)\right)\right) = 1$, the walk $\alpha^{(k)}(v_i, v_j)$ is called a positive walk, and negative walk otherwise. Denote by $w\left(\eta^{(k)}(v_i, v_j)\right)$ the total weight of a set of k-walks from v_i to v_j, which is defined as the sum of weighted k-walks in the set $\eta^{(k)}(v_i, v_j)$. Since a walk $\alpha(v_i, v_j)$ can have repetitive edges, the minimal walk between v_i and v_j is defined as the walk with the smallest k from v_i to v_j.

Example 1. To illustrate graph walks, Fig. 1 shows a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with node set $\mathcal{V} = \{v_1, v_2, v_3\}$, edge set $\mathcal{E} = \{(v_1, v_2), (v_2, v_3)\}$.

[1] Generalizations to multi-dimensional system states (e.g., $x_i \in \mathbb{R}^m$) are expected to be trivial via the matrix Kronecker product.

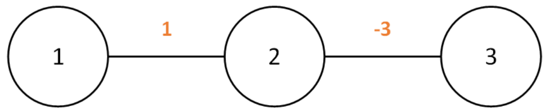

Figure 1. Graph walks on a weighted signed graph.

and the weighted adjacency matrix $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix}$. For example, the set of 3-walks $\eta^{(3)}(v_1, v_2) = \left\{\alpha_1^{(3)}(v_1, v_2), \alpha_2^{(3)}(v_1, v_2)\right\}$ consists of two walks between v_1 and v_2, where

$$\alpha_1^{(3)}(v_1, v_2) = \{v_1, (v_1, v_2), v_2, (v_2, v_3), v_3, (v_3, v_2), v_2\}$$

and

$$\alpha_2^{(3)}(v_1, v_2) = \{v_1, (v_1, v_2), v_2, (v_2, v_1), v_1, (v_1, v_2), v_2\}.$$

The weight of $\eta^{(3)}(v_1, v_2)$ can be computed as

$$\begin{aligned} w\left(\eta^{(3)}(v_{1,}v_2)\right) &= w\left(\alpha_1^{(3)}(v_1, v_2)\right) + w\left(\alpha_2^{(3)}(v_1, v_2)\right) \\ &= (1)^3 + (1) \times (-3)^2 = 10. \end{aligned}$$

As indicated in Lemma 1, network herdability depends on the existence of an element-wise positive vector $k \in \operatorname{range}(\mathcal{C})$, where $\mathcal{C}$ is the controllability matrix of $\mathcal{G}$. The key observation is that the columns of $\mathcal{C}$ are composed of the product of the input matrix B and the matrix $\mathcal{A}^k$, $k = 0, \ldots, n-1$. The following Lemma shows how graph walks relate to the powers of the adjacency matrix $\mathcal{A}$.

Lemma 2. *Consider a weighted signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{A}$ is the weighted adjacency matrix. Let $\mathcal{A}^k$ denote the kth power of $\mathcal{A}$. Each entry of $\mathcal{A}^k$ is determined as $\left[\mathcal{A}^k\right]_{ij} = w\left(\eta^{(k)}(v_{i,}v_j)\right)$, which is the sum of the weights of all k-walks from v_i to v_j.*

The proof is omitted here since it is a trivial extension of known properties of adjacency matrix [23]. Based on Lemma 2, when considering the product $\mathcal{A}^k B$, the entry $\left[\mathcal{A}^k B\right]_{ij}$ indicates the sum of the weights of all k-walks from node v_i to the leader v_j, $j \in \{1, \ldots, m\}$. This observation motivates the use of graph walks to characterize herdability of signed graphs based on the controllable subspace of $\mathcal{C}$.

B. Herdability on 1-Walk

The following theorem characterizes network herdability via 1-walk, which will then be extended to the cases of 2-walk and n-walks in the subsequent sections.

Theorem 1. *Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ evolving according to the dynamics in (1). The network $\mathcal{G}$ is herdable by the leader set $\mathcal{V}_l$, if the following two conditions hold: 1) there exists a 1-walk from each follower to at least one leader in $\mathcal{V}_l$, and 2) the 1-walks from followers to the same leader have the same sign.*

Proof: Consider a signed graph $\mathcal{G}$ with the weighted adjacency matrix $\mathcal{A}$ and the input matrix B. Since the first m nodes are assumed as the leaders, $B = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$ can be rewritten as $B = \begin{bmatrix} I_{m\times m} & 0_{m\times(n-m)} \end{bmatrix}^T$, where $I_{m\times m}$ is an m-dimensional identity matrix and $0_{(n-m)\times m}$ is a zero matrix. The system controllability matrix $\mathcal{C}$ is

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} B \mid \mathcal{A}B \mid \cdots & \mathcal{A}^{n-1}B \end{bmatrix} \\ &= \left[\begin{array}{c|c|c} I_{m\times m} & *_{m\times m} & \cdots \\ 0_{(n-m)\times m} & \Xi & \cdots \end{array}\right], \end{aligned} \tag{2}$$

where $*_{m\times m} \in \mathbb{R}^{m\times m}$ and $\Xi \in \mathbb{R}^{(n-m)\times m}$. Based on Lemma 1, the network $\mathcal{G}$ is herdable if and only if there exists an element-wise positive vector $k \in \operatorname{range}(\mathcal{C})$. That is, there exists a vector $\delta = \begin{bmatrix} \delta_1 & \ldots & \delta_{mn} \end{bmatrix}^T \in \mathbb{R}^{mn}$ such that $k = \begin{bmatrix} k_1 & \ldots & k_n \end{bmatrix}^T = \mathcal{C}\delta \in \mathbb{R}^n$ is element-wise positive. Note that the vector k is a linear combination of the columns of $\mathcal{C}$, i.e., $k = \sum_{i=1}^{mn} \delta_i \mathcal{C}_{:,i}$, where $\mathcal{C}_{:,i}$ represents the ith column of $\mathcal{C}$. The following proof will show that the first $2m$ columns of $\mathcal{C}$ are sufficient to guarantee the existence of the vector k, provided that the two conditions stated in the theorem are satisfied.

As indicated in (2), the first m columns of $\mathcal{C}$ correspond to the input matrix B, where the non-zero entries indicate direct access to the leader set $\mathcal{V}_l$, i.e., the set of nodes with 0-walk to $\mathcal{V}_l$. Due to the identity matrix $I_{m\times m}$, there always exist sufficiently large $\delta_i \in \mathbb{R}^+$, $i = 1, \ldots, m$, such that $\delta_i > \sum(\mathcal{C}_{i,i+1:mn})$, where $\sum(\mathcal{C}_{i,i+1:mn})$ represents the sum of the entries from the$(i+1)$th column to the mnth column in the ith row in (2). Therefore, the first m entries in k are guaranteed to be positive if δ_i, $i = 1, \ldots, m$, are selected sufficiently large.

The second m columns of $\mathcal{C}$ correspond to $\mathcal{A}B$. Based on Lemma 2, each non-zero entry Ξ_{ij} indicates there exists a 1-walk from the follower v_{m+i} to the leader $v_j \in \mathcal{V}_l$. If there exists a 1-walk from each follower to at least one leader in $\mathcal{V}_l$, then no rows in Ξ are zeros. In addition, if the followers connected to the same leader via 1-walk have the same signs, the nonzero entries in each column of Ξ have the same sign (i.e., either all positive or all negative). Then there always exists a proper design of δ_i, $i = m+1, \ldots, 2m$, such that the linear combination of the columns of Ξ is elementary positive.

Based on the analysis above, with the design of the rest entries of δ as $\delta_i = 0$, $i = 2m+1, \ldots mn$, there exists an element-wise positive vector $k \in \operatorname{range}(\mathcal{C})$, which indicates the network $\mathcal{G}$ is herdable. ■

Theorem 1 characterizes a class of herdable networks. For these networks, the analysis above indicates that the herdability of $\mathcal{G}$ depends on the matrix B (i.e., 0-walks to the leader set) and $\mathcal{A}B$ (i.e., 1-walks from the followers to the leader set). In other words, follower-to-follower and leader-to-leader connections will not affect the herdability of the system. That is, a network remains herdable, regardless of the addition and removal of follower-to-follower and leader-to-leader connections, or the change of edge signs.

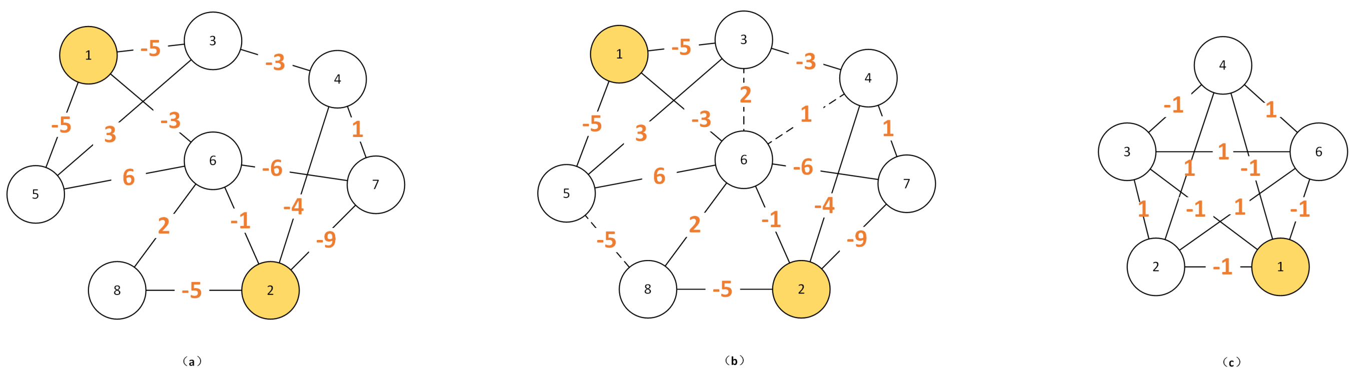

Figure 2. Examples of herdable signed graphs.

When considering a single leader, the following corollary is a special case of Theorem 1, where the leader set $\mathcal{V}_l$ is a singleton.

Corollary 1. *Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ evolving according to the dynamics in (1). The network $\mathcal{G}$ is herdable by a single leader if there exists a 1-walk from each follower to the leader and leader-follower connections have the same edge signs.*

As demonstrated in [6], a complete graph is not fully controllable under a single leader, due to the topological symmetry with respect to the leader. In contrast, Corollary 1 indicates that a complete graph can be herdable, if the leader-follower connections have the same sign. To illustrate Theorem 1 and Corollary 1, the following examples are provided.

Example 2. Fig. 2 (a) and (b) consider a weighted signed network $\mathcal{G}$ with the leader set $\mathcal{V}_l = \{v_1, v_2\}$. Based on Theorem 1, Fig. 2 (a) is herdable, since there exists a 1-walk from each follower to at least one leader in $\mathcal{V}_l$, and the 1-walks from followers to the same leader have the same sign. Fig 2 (b) is obtained from Fig. 2 (a) by adding new follower-to-follower connections (the dashed lines). It can be verified that Fig 2 (b) remains herdable. Figure 2 (c) is an example on complete graph, which is not controllable, but herdable by the leader v_1.

C. Herdability on 2-walk

This section extends Theorem 1 and characterizes the herdability of a class of networks where the followers are reachable from the leader set within 2-walks. Suppose the nodes can be classified into two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}_f$, where the nodes in $\mathcal{V}_1$ and $\mathcal{V}_2$ are reachable via 1-walks and 2-walks from at least one leader in $\mathcal{V}_l$, respectively. Let $\mathcal{V}_{12}$ denote the set of nodes that are reachable via both 1-walks and 2-walks. Note that $\mathcal{V}_1$ and $\mathcal{V}_2$ are not necessarily mutual exclusive and some followers can be reachable from the leader set via both 1-walk and 2-walk. Let $\mathcal{V}_{12} = \mathcal{V}_1 \cap \mathcal{V}_2$ denote such a set of nodes.

Theorem 2. *Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ evolving according to the dynamics in (1). The network $\mathcal{G}$ is herdable by the leaders $\mathcal{V}_l$, if the following conditions hold: 1) the nodes in $\mathcal{V}_f$ are reachable by at least one leader via either a 1-walk or a 2-walk, i.e.,$\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}_f$; 2) the 1-walks and the 2-walks from the followers to the same leader in $\mathcal{V}_l$ have the same sign, respectively.*

Proof: Based on the weighted adjacency matrix $\mathcal{A}$ and the input matrix B, the system controllability matrix $\mathcal{C}$ can be written as

$$\begin{aligned}\mathcal{C} &= \left[\begin{array}{c|c|c} B & \mathcal{A}B & \mathcal{A}^2 B \quad \cdots \end{array}\right] \\ &= \left[\begin{array}{c|c|c|c} I_{m\times m} & *_{m\times m} & *_{m\times m} & \cdots \\ 0_{(n-m)\times m} & \Psi & \Phi & \cdots \end{array}\right],\end{aligned} \tag{3}$$

where $*_{m\times m} \in \mathbb{R}^{m\times m}$ and $\Psi, \Phi \in \mathbb{R}^{(n-m)\times m}$. Assume that there are q nodes in $\mathcal{V}_1$, p nodes in $\mathcal{V}_2$, and r nodes in $\mathcal{V}_{12}$ with $p + q \geq n - m$ and $r \leq \min\{p, q\}$. Without loss of generality, the nodes indices are reordered as $v_i \in \mathcal{V}_1$, $i = m+1, \ldots, m+q$ and $v_j \in \mathcal{V}_2$, $j = n-p, \ldots, n$. It is further assumed that the first r nodes in $\mathcal{V}_1$ also belong to $\mathcal{V}_{12}$, i.e., $v_i \in \mathcal{V}_{12}$, $i = m+1, \ldots, m+r$. Based on the defined $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_{12}$, the matrix Ψ and Φ in (3) can be rewritten as

$$\Psi = \begin{bmatrix} \Psi^{(1)}_{q\times m} \\ 0_{(n-m-q)\times m} \end{bmatrix} \text{ and } \Phi = \begin{bmatrix} \Phi^{(12)}_{r\times m} \\ 0_{(n-m-p-r)\times m} \\ \Phi^{(2)}_{p\times m} \end{bmatrix}, \tag{4}$$

where the non-zero entry (i, j) in $\Psi^{(1)}_{q\times m}$ indicates that there exists a 1-walk from the follower v_{m+i} to the leader v_j. Similarly, the non-zero entry (i, j) in $\Phi^{(12)}_{r\times m}$ indicates the follower v_{m+i} are reachable from the leader v_j via both 1-walks and 2-walks, while the non-zero entry (i, j) in $\Phi^{(2)}_{p\times m}$ indicates the follower v_{n-p+i} are reachable from the leader v_j via only 2-walks.

To show that the network $\mathcal{G}$ is herdable by the leader set $\mathcal{V}_l$, based on Lemma 1, we need to show there exists a vector $\delta = \left[\begin{array}{ccc} \delta_1 & \ldots & \delta_{mn} \end{array}\right]^T \in \mathbb{R}^{mn}$ such that $k = \left[\begin{array}{ccc} k_1 & \ldots & k_n \end{array}\right]^T = \mathcal{C}\delta \in \mathbb{R}^n$ is element-wise positive. The vector k is a linear combination of the columns of $\mathcal{C}$ with respect to δ. The following proof will show that the first $3m$

columns of $\mathcal{C}$ are sufficient to guarantee the existence of the vector k, provided that the conditions stated in the theorem are satisfied.

First, following similar analysis as in the proof of Theorem 1, it is straightforward to verify that, due to the identity matrix $I_{m\times m}$, there always exist sufficiently large $\delta_i \in \mathbb{R}^+$, $i = 1, \ldots, m$, such that the first m entries of the linear combination of the columns in (3) are positive. That is, the first m entries in k are guaranteed to be positive if δ_i, $i = 1, \ldots, m$, are selected sufficiently large.

The second and third m columns of $\mathcal{C}$ correspond to $\mathcal{A}B$ and $\mathcal{A}^2B$. If the followers connected to the same leader via 1-walks have the same edge signs, the nonzero entries in each column of Ψ have the same sign (i.e., either all positive or all negative). Similar argument indicates that the nonzero entries in each column of Φ also have the same sign. In addition, since each follower is reachable from the leaders via at least one leader, no rows in (3) are all zeros. Therefore, there always exists a proper design of δ_i, $i = m+1, \ldots, 2m$, such that the linear combination of the columns of Ψ and Φ is element-wise positive.

Based on the analysis above, with the design of $\delta_i = 0$, $i = 3m+1, \ldots mn$, there exists an element-wise positive vector $k \in \text{range}(\mathcal{C})$, which indicates the network $\mathcal{G}$ is herdable. ■

Theorem 2 extends Theorem 1 by characterizing a class of herdable networks that allows followers reachable from the leader set via 1-walks and 2-walks.

D. Herdability of Acyclic Graphs via Multiple Walks

Section III-B and III-C characterize the herdability of general signed weighted graphs with 1-walks and 2-walks. Extension of the developed results in Section III-B and III-C to networks with multiple walks is challenging in general, mainly due to the increasing volume of walks and the existence of cyclic walks between network nodes. As a first attempt, this section considers a particular class of graphs, namely acyclic graphs, and characterizes its herdability under a single leader with multiple walks. Acyclic graph is a class of graphs without cycles. Path and tree graphs are typical examples of acyclic graphs.

Proposition 1. *Consider an acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Let $\mathcal{V}_o$ and $\mathcal{V}_e$ denote the set of nodes reachable from a leader set through $2N$-walks and $(2N+1)$-walks, $N = 0, 1, 2, \ldots$, respectively. Then one has $\mathcal{V}_o \cup \mathcal{V}_e = \mathcal{V}$ and $\mathcal{V}_o \cap \mathcal{V}_e = \emptyset$.*

Since acyclic graphs do not have cycles and any two nodes are connected via a unique path, Proposition 1 is an immediate consequence. Based on Proposition 1, the following theorem characterizes the herdability of signed acyclic graphs.

Theorem 3. *Consider a signed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ evolving according to the dynamics in (1). The graph $\mathcal{G}$ is herdable by a single leader v_l if the the walks from the nodes $v \in \mathcal{V}_o$ and from $v \in \mathcal{V}_e$ to v_l have the same sign, respectively.*

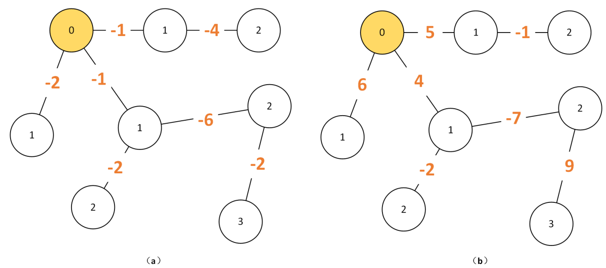

Figure 3. Examples of herdable trees under a single leader.

Proof: When considering a single leader, the input matrix B in (1) is reduced to a basis vector $e_l \in \mathbb{R}^n$, where the lth entry is 1 while the others are zeros. For the controllability matrix $\mathcal{C} \in \mathbb{R}^{n\times n}$, the last two columns are $\mathcal{A}^{n-2}e_l$ and $\mathcal{A}^{n-1}e_l$, where the nonzero entries in the lth column of $\mathcal{A}^{n-2}e_l$ and $\mathcal{A}^{n-1}e_l$ indicates the corresponding nodes are reachable from v_l via $(n-2)$-walks and $(n-1)$-walks, respectively. Note that repetitive edges are allowed in walks, which indicates that any node on an acyclic graph can reach v_l by either a $(n-1)$-walks or $(n-2)$-walks. Let $\mathcal{V}_o$ and $\mathcal{V}_e$ denote the set of nodes whose corresponding entries are non-zero in $\mathcal{A}^{n-2}e_l$ and $\mathcal{A}^{n-1}e_l$.

Based on Proposition 1, $\mathcal{V}_o \cup \mathcal{V}_e = \mathcal{V}$ and $\mathcal{V}_o \cap \mathcal{V}_e = \emptyset$, which indicates the indices of non-zero entries of $\mathcal{A}^{n-2}e_l$ and $\mathcal{A}^{n-1}e_l$ are mutually exclusive. Therefore, if the non-zero entries of $\mathcal{A}^{n-2}e_l$ and $\mathcal{A}^{n-1}e_l$ have the same signs, respectively, then there always exists a vector $\delta = \begin{bmatrix} \delta_1 & \ldots & \delta_n \end{bmatrix}^T \in \mathbb{R}^n$ such that

$$k = \mathcal{A}^{n-2}e_l\delta_{n-1} + \mathcal{A}^{n-1}e_l\delta_n \in \mathbb{R}^n$$

is element-wise positive. One example design of δ is $\delta_i = 0$, $i = 1, \ldots, n-3$, and δ_{n-2} and δ_{n-1} are selected according to the sign of the entries in $\mathcal{A}^{n-2}e_l$ and $\mathcal{A}^{n-1}e_l$. The existence of $k \in \text{range}(\mathcal{C})$ indicates the herdability based on Lemma 1. ■

Examples are provided to illustrate Theorem 3.

Example 3. Fig. 3 (a) and (b) show two examples of herdable trees under a single leader. The filled node indexed 0 represents the leader v_l, while the rest unfilled nodes represent the followers. The number on each follower represents the walks from leader to the follower. In Fig. 3 (a), the walks from v_l to all nodes in $v \in \mathcal{V}_o$ (i.e. the set of nodes with odd number of walks to the leader) all have negative signs, while the walks from v_l to all nodes in $v \in \mathcal{V}_e$ (i.e. the set of nodes with even number of walks to the leader) all have positive signs. According to Theorem 3, the tree in Fig. 3 (a) is herdable. Fig. 3 (b) is an additional example, which shows, if the edge signs are flipped, the tree remains herdable according to Theorem 3.

Remark 1. The work of [1] compares the controllability and herdability over networks whose topological structures are symmetric with respect to a single leader (i.e., graphs with leader symmetry). It is discovered that positive networks are not controllable due to the leader symmetry. However, such positive networks are indeed herdable. Different from positive networks, signed networks with leader symmetry are not guaranteed to be herdable.

IV. Discussion

The herdability of weighted signed graphs considered in this work can be extended to sign herdability. The concept of sign herdability is originated from the sign controllability introduced in [24] and [25]. Both sign controllability and sign herdability focus on characterizing the controllability or herdability of a class of graphs by investigating its sign pattern. Specifically, provided a weighted signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, its corresponding sign pattern is defined as $\mathcal{G}_s = (\mathcal{V}, \mathcal{E}, \mathcal{A}_s)$, where the node set and edge set remains the same. The difference is the edge weights in $\mathcal{G}$ are abstracted to only "+" and "-" signs, i.e., $[\mathcal{A}_s]_{ij} = " + "$ if $[\mathcal{A}]_{ij} > 0$ and $[\mathcal{A}_s]_{ij} = " - "$ if $[\mathcal{A}]_{ij} < 0$. Therefore, the sign pattern $\mathcal{G}_s$ can represent a class of graphs.

Networked systems represented by graph $\mathcal{G}$ is sign equivalent if they share the same sign pattern $\mathcal{G}_s$. Consider a sign pattern $\mathcal{G}_s = (\mathcal{V}, \mathcal{E}, \mathcal{A}_s)$. If $(\mathcal{A}_s, B)$ is herdable by a leader set, any network featured of the same sign pattern $\mathcal{G}_s$ and B is herdable. That is, any graphs sign equivalent to $\mathcal{G}_s$ and B have the same herdability. Note that the characterizations of herdability developed in this work are only based on the sign of edges. Therefore, the developed results indeed characterize the herdability of sign patterns. The major benefit of considering herdability in terms of sign patterns is its robustness to system parametric uncertainties. Since systems are prone to modeling errors or disturbances, traditional analysis of herdability are sensitive to the values of system matrices A and B. In contrast, the analysis over sign patterns is only characterized based on the edge signs, regardless of the magnitude of the edge weights. Therefore, the herdability can be characterized based on only qualitative information about A and B (e.g., the signs of their entries).

V. Conclusion

Herdability of weighted signed graphs is considered in this work, where graph walks are exploited to characterize topological structures that render network herdability. Sufficient conditions ensuring the herdability of signed networks via 1-walks and 2-walks are developed, which are then extended to characterize acyclic graphs with multiple walks. Additional research will consider more general topological structures (e.g., directed graphs with non-symmetric adjacency matrices) and investigate the underlying relationship between network topologies and herdability.

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